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UMBRAE FOR DEGENERATE BERNOULLI AND DEGENERATE EULER NUMBERS

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ABSTRACT. In this note, we introduce symbolical notations, called umbrae, for degenerate Bernoulli and degenerate Euler numbers. Umbrae have been used, for example, in connection with Bernoulli and Euler numbers. We derive some identities, properties and relations for the degenerate Bernoulli and degenerate Euler numbers and polynomials by using those umbrae.

1. INTRODUCTION

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are given by

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\quad e_\lambda(t)=e_\lambda^1(t),\quad (\text{see } [1-3,5-10]), \tag{1}$$

where $(x)_{0,\lambda}=1,\ (x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots\big(x-(n-1)\lambda\big),\ (n\geq 1)$.

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (\text{see } [3,5-10]), \tag{2}$$

and the Bernoulli numbers B_n by $B_n=B_n(0)$.
The Euler polynomials $E_n(x)$ are given by

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad (\text{see } [9]), \tag{3}$$

and the Euler numbers E_n by $E_n=E_n(0)$.

In [1,2], Carlitz introduced the degenerate Bernoulli polynomials $\beta_{n,\lambda}(x)$ defined by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see } [1-3,5-10]). \tag{4}$$

For $x=0$, $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.
In addition, the degenerate Euler polynomials $\mathscr{E}_{n,\lambda}(x)$ are given by

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see } [9]). \tag{5}$$

For $x=0$, $\mathscr{E}_{n,\lambda}=\mathscr{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers.
Note here that $\lim_{\lambda\to 0}\beta_{n,\lambda}(x)=B_n(x)$, and $\lim_{\lambda\to 0}\mathscr{E}_{n,\lambda}(x)=E_n(x)$.

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In this note, we introduce symbolical notations, called umbrae, for the Carlitz's degenerate Bernoulli and degenerate Euler numbers. For instance, umbrae have been used for the Bernoulli and Euler numbers. We derive some identities, properties and relations for the degenerate Bernoulli and degenerate Euler numbers and polynomials by making use of those umbrae.

2. Umbrae for degenerate Bernoulli and degenerate Euler numbers

As to (4), we introduce the umbra β (see [4]) for the degenerate Bernoulli numbers as follows:

$$\frac{t}{e_\lambda(t)-1}=e_\lambda^{\beta}(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{t^n}{n!}, \tag{6}$$

with the convention about replacing $(\beta)_{n,\lambda}$ by $\beta_{n,\lambda}$, $(n\geq 0)$.

By (6), we get

$$t=e_\lambda^{\beta+1}(t)-e_\lambda^{\beta}(t)=\sum_{n=0}^{\infty}\Big((\beta+1)_{n,\lambda}-(\beta)_{n,\lambda}\Big)\frac{t^n}{n!}. \tag{7}$$

Thus, by comparing the coefficients on both sides of (7), we have

$$(\beta+1)_{n,\lambda}-(\beta)_{n,\lambda}=\begin{cases}1, & \text{if } n=1,\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

with the above-mentioned convention about replacing $(\beta)_{n,\lambda}$ by $\beta_{n,\lambda}$, $(n\geq 0)$.

We see from (1) that

$$(x+y)_{n,\lambda}=\sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}(y)_{n-k,\lambda},\quad (n\geq 0). \tag{9}$$

By (8) and (9), for any positive integer n, we have

$$\begin{aligned}0=(\beta+1)_{n+1,\lambda}-\beta_{n+1,\lambda}&=\sum_{k=0}^{n}\binom{n+1}{k}\beta_{k,\lambda}(1)_{n+1-k,\lambda}\\ &=(n+1)\beta_{n,\lambda}+\sum_{k=0}^{n-1}\binom{n+1}{k}\beta_{k,\lambda}(1)_{n+1-k,\lambda}.\end{aligned} \tag{10}$$

By (10), we get

$$\beta_{n,\lambda}=-\frac{1}{(n+1)}\sum_{k=0}^{n-1}\binom{n+1}{k}\beta_{k,\lambda}(1)_{n+1-k,\lambda}. \tag{11}$$

By (6), the degenerate Bernoulli polynomials are written as

$$\frac{t}{e_\lambda(t)-1}e_\lambda^{x}(t)=e_\lambda^{\beta+x}(t)=\sum_{n=0}^{\infty}(\beta+x)_{n,\lambda}\frac{t^n}{n!}. \tag{12}$$

From (4), (9) and (12), we note that

$$\begin{aligned}\beta_{n,\lambda}(x)=(\beta+x)_{n,\lambda}&=\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}(x)_{n-k,\lambda}\\ &=\sum_{k=0}^{n}\binom{n}{k}\beta_{n-k,\lambda}(x)_{k,\lambda},\quad (n\geq 0).\end{aligned} \tag{13}$$

Now, we observe that

$$(14)\quad \sum_{m=0}^{\infty}\sum_{k=0}^{n-1}(k+x)_{m,\lambda}\frac{t^m}{m!}=\sum_{k=0}^{n-1}e_{\lambda}^{k+x}(t)=\frac{1}{t}\frac{t}{e_{\lambda}(t)-1}\Big(e_{\lambda}^{n+x}(t)-e_{\lambda}^{x}(t)\Big)$$
$$=\frac{1}{t}\Big(e_{\lambda}^{\beta+n+x}(t)-e_{\lambda}^{\beta+x}(t)\Big)$$
$$=\sum_{m=0}^{\infty}\frac{1}{m+1}\Big((\beta+n+x)_{m+1,\lambda}-(\beta+x)_{m+1,\lambda}\Big)\frac{t^m}{m!}.$$

By (13) and comparing the coefficients on both sides of (14), we get

$$(15)\quad \sum_{k=0}^{n-1}(k+x)_{m,\lambda}=\frac{1}{m+1}\Big((\beta+x+n)_{m+1,\lambda}-(\beta+x)_{m+1,\lambda}\Big)$$
$$=\frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}(\beta+x)_{l,\lambda}(n)_{m+1-l,\lambda}$$
$$=\frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}\beta_{l,\lambda}(x)(n)_{m+1-l,\lambda}.$$

As to (5), we introduce the umbra $\mathscr{E}$ (see [4]) for the degenerate Euler numbers as follows:

$$(16)\quad \frac{2}{e_{\lambda}(t)+1}=e_{\lambda}^{\mathscr{E}}(t)=\sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}\frac{t^n}{n!},$$

with the convention about replacing $(\mathscr{E})_{n,\lambda}$ by $\mathscr{E}_{n,\lambda}$, $(n\geq 0)$.
Then the degenerate Euler polynomials can be written as

$$(17)\quad \frac{2}{e_{\lambda}(t)+1}e_{\lambda}^{x}(t)=e_{\lambda}^{\mathscr{E}+x}(t)=\sum_{n=0}^{\infty}\big(\mathscr{E}+x\big)_{n,\lambda}\frac{t^n}{n!}.$$

Thus, by (5), (9) and (17), we get

$$(18)\quad \mathscr{E}_{n,\lambda}(x)=\big(\mathscr{E}+x\big)_{n,\lambda}=\sum_{k=0}^{n}\binom{n}{k}\mathscr{E}_{k,\lambda}(x)_{n-k,\lambda}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\mathscr{E}_{n-k,\lambda}(x)_{k,\lambda},\quad (n\geq 0).$$

Note from (16) that

$$(19)\quad 2=e_{\lambda}^{\mathscr{E}}(t)\big(e_{\lambda}(t)+1\big)=e_{\lambda}^{\mathscr{E}+1}(t)+e_{\lambda}^{\mathscr{E}}(t)=\sum_{n=0}^{\infty}\Big((\mathscr{E}+1)_{n,\lambda}+\mathscr{E}_{n,\lambda}\Big)\frac{t^n}{n!}.$$

Thus, by (19), we get

$$(20)\quad (\mathscr{E}+1)_{n,\lambda}+\mathscr{E}_{n,\lambda}=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n\geq 1.\end{cases}$$

For $n\geq 1$, by (20), we obtain

$$(21)\quad \mathscr{E}_{n,\lambda}=-\frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{E}_{k,\lambda}(1)_{n-k,\lambda}.$$

By (20) and (21), we get

$$\mathscr{E}_{0,\lambda}=1,\ \mathscr{E}_{1,\lambda}=-\frac{1}{2},\ \mathscr{E}_{2,\lambda}=-\frac{1}{2}(1)_{2,\lambda}+\frac{1}{2},$$
$$\mathscr{E}_{3,\lambda}=-\frac{1}{2}(1)_{3,\lambda}+\frac{3}{2}(1)_{2,\lambda}-\frac{3}{4},\cdots.$$

Now, we observe that

$$2\sum_{k=0}^{n-1}(-1)^k e_\lambda^k(t)=\frac{2}{e_\lambda(t)+1}+\frac{2}{e_\lambda(t)+1}e_\lambda^n(t)=e_\lambda^{\mathscr{E}}(t)+e_\lambda^{\mathscr{E}+n}(t)$$
$$=\sum_{m=0}^{\infty}\Big(\mathscr{E}_{m,\lambda}+(\mathscr{E}+n)_{m,\lambda}\Big)\frac{t^m}{m!},$$

where $n\in\mathbb{N}$ with $n\equiv 1\,(\mathrm{mod}\ 2)$.
Thus, we have

$$\sum_{k=0}^{n-1}(-1)^k(k)_{m,\lambda}=\frac{1}{2}\big(\mathscr{E}_{m,\lambda}+(\mathscr{E}+n)_{m,\lambda}\big)$$
$$=\frac{1}{2}\big(\mathscr{E}_{m,\lambda}+\mathscr{E}_{m,\lambda}(n)\big),$$

where $n\in\mathbb{N}$ with $n\equiv 1\,(\mathrm{mod}\ 2)$.
Also, for $n\in\mathbb{N}$ with $n\equiv 0\,(\mathrm{mod}\ 2)$, we have

$$\sum_{k=0}^{n-1}(-1)^k(k)_{m,\lambda}=\frac{1}{2}\big(\mathscr{E}_{m,\lambda}-(\mathscr{E}+n)_{m,\lambda}\big)$$
$$=\frac{1}{2}\big(\mathscr{E}_{m,\lambda}-\mathscr{E}_{m,\lambda}(n)\big),\quad (n\geq 0).$$

For $d\in\mathbb{N}$, we have

$$(22)\qquad e_\lambda^{\beta+x}(t)=\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=t\sum_{k=0}^{d-1}\frac{e_\lambda^{k+x}(t)}{e_\lambda^d(t)-1}$$
$$=\frac{1}{d}\sum_{k=0}^{d-1}\frac{e_{\frac{\lambda}{d}}^{\frac{k+x}{d}}(dt)dt}{e_{\frac{\lambda}{d}}(dt)-1}=\frac{1}{d}\sum_{k=0}^{d-1}e_{\frac{\lambda}{d}}^{\beta+\frac{k+x}{d}}(dt).$$

From (22), we note that

$$(23)\qquad \sum_{n=0}^{\infty}(\beta+x)_{n,\lambda}\frac{t^n}{n!}=\frac{1}{d}\sum_{n=0}^{\infty}\sum_{k=0}^{d-1}\Big(\beta+\frac{k+x}{d}\Big)_{n,\frac{\lambda}{d}}\frac{d^nt^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(d^{n-1}\sum_{k=0}^{d-1}\Big(\beta+\frac{k+x}{d}\Big)_{n,\frac{\lambda}{d}}\right)\frac{t^n}{n!}.$$

By comparing the coefficients on both sides of (23), we get

$$(24)\qquad (\beta+x)_{n,\lambda}=d^{n-1}\sum_{k=0}^{d-1}\Big(\beta+\frac{k+x}{d}\Big)_{n,\frac{\lambda}{d}},\quad (n\geq 0).$$

From (13) and (24), we have

$$\beta_{n,\lambda}(x)=d^{n-1}\sum_{k=0}^{d-1}\beta_{n,\frac{\lambda}{d}}\Big(\frac{k+x}{d}\Big),\quad (n\geq 0,\ d\in\mathbb{N}). \tag{25}$$

For $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$, we have

$$\begin{aligned} e_{\lambda}^{\mathscr{E}+x}(t)&=\frac{2e_{\lambda}^{x}(t)}{e_{\lambda}^{d}(t)+1}\sum_{k=0}^{d-1}(-1)^{k}e_{\lambda}^{k}(t)\\ &=\sum_{k=0}^{d-1}(-1)^{k}\frac{2}{e_{\frac{\lambda}{d}}(dt)+1}e_{\frac{\lambda}{d}}^{\frac{k+x}{d}}(dt)=\sum_{k=0}^{d-1}(-1)^{k}e_{\frac{\lambda}{d}}^{\mathscr{E}+\frac{k+x}{d}}(dt)\\ &=\sum_{n=0}^{\infty}\bigg(d^{n}\sum_{k=0}^{d-1}(-1)^{k}\Big(\mathscr{E}+\frac{k+x}{d}\Big)_{n,\frac{\lambda}{d}}\bigg)\frac{t^{n}}{n!}. \end{aligned} \tag{26}$$

Thus, by (26), we get

$$(\mathscr{E}+x)_{n,\lambda}=d^{n}\sum_{k=0}^{d-1}(-1)^{k}\Big(\mathscr{E}+\frac{k+x}{d}\Big)_{n,\frac{\lambda}{d}},\quad (n\geq 0). \tag{27}$$

Equivalently, (27) is equal to

$$\mathscr{E}_{n,\lambda}(x)=d^{n}\sum_{k=0}^{d-1}(-1)^{k}\mathscr{E}_{n,\frac{\lambda}{d}}\Big(\frac{k+x}{d}\Big),\quad (n\geq 0). \tag{28}$$

Let $d\in\mathbb{N}$ with $d\equiv 0 \pmod 2$. Then we have

$$\begin{aligned} \frac{2}{e_{\lambda}(t)+1}e_{\lambda}^{x}(t)&=\sum_{k=0}^{d-1}(-1)^{k-1}e_{\lambda}^{k+x}(t)\frac{2}{e_{\lambda}^{d}(t)-1}\\ &=\frac{2}{dt}\sum_{k=0}^{d-1}(-1)^{k-1}e_{\frac{\lambda}{d}}^{\frac{k+x}{d}}(dt)\frac{dt}{e_{\frac{\lambda}{d}}(dt)-1}\\ &=\frac{2}{dt}\sum_{k=0}^{d-1}(-1)^{k-1}e_{\frac{\lambda}{d}}^{\beta+\frac{k+x}{d}}(dt). \end{aligned} \tag{29}$$

By multiplying (29) by t on both sides, we obtain

$$\frac{2}{d}\sum_{k=0}^{d-1}(-1)^{k-1}e_{\frac{\lambda}{d}}^{\beta+\frac{k+x}{d}}(dt)=te_{\lambda}^{\mathscr{E}+x}(t)=\sum_{n=1}^{\infty}n(\mathscr{E}+x)_{n-1,\lambda}\frac{t^{n}}{n!}. \tag{30}$$

By (13), (18) and (30), we get

$$\mathscr{E}_{n-1,\lambda}(x)=\frac{2}{n}d^{n-1}\sum_{k=0}^{d-1}(-1)^{k-1}\beta_{n,\frac{\lambda}{d}}\Big(\frac{k+x}{d}\Big),$$

where $n\in\mathbb{N}$, and $d\in\mathbb{N}$ with $d\equiv 0 \pmod 2$.

3. Conclusion

In connection with the Bernoulli and Euler numbers, the following umbrae B and E have been used (see [4]):

$$\frac{t}{e^t-1}=e^{Bt}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},$$
$$\frac{2}{e^t+1}=e^{Et}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},$$

with the conventions about replacing B^n by B_n, and E^n by E_n.

In this note, we introduced the umbrae β and $\mathscr{E}$, respectively for the degenerate Bernoulli and degenerate Euler numbers (see (6), (16)) and showed some of their uses. We note that, by taking $\lambda \to 0$, and replacing β and $\mathscr{E}$, respectively by B and E, we get the corresponding results for the Bernoulli and Euler numbers and polynomials. In other words, our results are degenerate versions of some of the well-known identities, properties and relations for the Bernoulli and Euler numbers and polynomials.

References

[1] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51-88.

[2] Carlitz, L. *A degenerate Staudt-Clausen theorem,* Arch. Math. (Basel) **7** (1956), 28-33.

[3] Chung, S.-K.; Jang, G.-W.; Kim, D. S.; Kwon, J. *Some identities of the type 2 degenerate Bernoulli and Euler numbers,* Adv. Stud. Contemp. Math. (Kyungshang) **29** (2019), no. 4, 613-632.

[4] Guinand, A. P. *The umbral method: A survey of elementary mnemonic and manipulative uses,* Amer. Math. Monthly **86** (1979), no. 3, 187-195.

[5] Khan, W. A.; Ahmad, M. *Partially degenerate poly-Bernoulli polynomials associated with Hermite polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **28** (2018), no. 3, 487-496.

[6] Kim, D. S.; Kim, T.; Dolgy, D. V.; Komatsu, T. *Barnes-type degenerate Bernoulli polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **25** (2015), no. 1, 121-146.

[7] Kim, T.; Kim, D. S.; Kwon, H.-I. *Some identities of Carlitz degenerate Bernoulli numbers and polynomials,* Iran. J. Sci. Technol. Trans. A Sci. **41** (2017), no. 3, 749-753.

[8] Kim, W. J.; Jang, L.-C.; Kwon, J. *Some properties of generalized degenerate Bernoulli polynomials and numbers,* Adv. Stud. Contemp. Math. (Kyungshang) **32** (2022), no. 4, 479-486.

[9] Luo, L.; Kim, T.; Kim, D. S.; Ma, Y. *Probabilistic degenerate Bernoulli and degenerate Euler polynomials,* Math. Comput. Model. Dyn. Syst. **30** (2024), no. 1, 342-363.

[10] Park, J.-W.; Pyo, S.-S. *A note on degenerate Bernoulli polynomials arising from umbral calculus,* Adv. Stud. Contemp. Math. (Kyungshang) **32** (2022), no. 4, 509-525.

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